1) a. $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

Ratio test: $\lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{(2(n+1))} / (2(n+1))!}{(-1)^n x^{2n} / (2n)!} \right| = \lim_{n\to\infty} \left| \frac{x^{2n+2}}{x^{2n}} \frac{(2n)!}{(2n+2)!} \right|$

$= \lim_{n\to\infty} \left| x^2 / (n+1)(n+2) \right| = 0$

so $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ converges everywhere

b. $\tan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)}$

Root test: $\lim_{n\to\infty} \left| \frac{(-1)^n x^{2n+1}}{(2n+1)} \right|^{1/n} = \lim_{n\to\infty} \left| \frac{x^{2n+1}}{(2n+1)} \right|^{1/n} = \lim_{n\to\infty} \frac{|x^{2+1/n}|}{(2n+1)^{1/n}}$

$= \lim_{n\to\infty} \frac{x^2 |x|^{1/n}}{(2n+1)^{1/n}} = x^2$

Converges for $-1 < x < 1$, diverges for $x < -1$, $1 < x$.

Check $x=1$: $\sum_{n=0}^{\infty} (-1)^n / (2n+1)$ converges (alternating harmonic)

Check: $x=-1$ $\sum_{n=0}^{\infty} \frac{(-1)^n (-1)^{2n+1}}{(2n+1)} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)}$ diverges (compare to $\sum_{n=1}^{\infty} \frac{1}{3n}$)

so $\tan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n+1)}$ for $-1 < x \leq 1$

2) $\sum_{n=2}^{\infty} \frac{x^n}{3^n \ln(n)}$ Root test: $\lim_{n\to\infty} \left| \frac{x^n}{3^n} \frac{1}{\ln(n)} \right|^{1/n} = \lim_{n\to\infty} \left| \frac{x}{3} \right| (\ln(n))^{1/n} = \left| \frac{x}{3} \right|$

Converges for $-1 < \frac{x}{3} < 1$ or equivalently, $-3 < x < 3$

For $x=3$, have $\sum_{n=2}^{\infty} \frac{3^n}{3^n \ln(n)} = \sum_{n=2}^{\infty} \frac{1}{\ln(n)}$ diverges (compare to $\sum_{n=1}^{\infty} \frac{1}{n}$)

For $x=-3$, have $\sum_{n=2}^{\infty} \frac{(-3)^n}{3^n \ln(n)} = \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$ converges (alternating series test)

$\sum_{n=2}^{\infty} \frac{x^n}{3^n \ln(n)}$ converges for $-3 \leq x < 3$

3) $\sum_{n=0}^{\infty} n!\, x^n$

Root test: $\lim_{n\to\infty} \left| \frac{(n+1)!\, x^{n+1}}{n!\ \ x^n} \right| = \lim_{n\to\infty} |(n+1)x| = \begin{cases} 0 & x=0 \\ \text{diverges} & x\neq 0 \end{cases}$

$\sum_{n=0}^{\infty} n!x^n$ converges for $x=0$

4) $\sum_{n=0}^{\infty} \frac{(x-3)^n}{n^2+1}$

Ratio test: $\lim_{n\to\infty} \left| \frac{(x-3)^n}{n^2+1} \right|^{1/n} = \lim_{n\to\infty} \frac{|x-3|}{(n^2+1)^{1/n}} = |x-3|$

converges for $-1 < x-3 < 1$ or $2 < x < 4$

Check $x=2$: $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$ converges (alternating series test)

Check $x=4$: $\sum_{n=0}^{\infty} \frac{1^n}{n^2+1}$ converges (compare to p-series $\sum_{n=1}^{\infty} \frac{1}{n^2} + 1$)

5) $1 - \ln(2) + \frac{(\ln(2))^2}{2} - \frac{(\ln(2))^3}{6} + \frac{(\ln(2))^4}{4!} - \dots = \sum_{n=0}^{\infty} \frac{(-\ln(2))^n}{n!}$

$= e^{-\ln(2)} = e^{\ln(1/2)} = 1/2.$

6) a. $\lim_{x\to 0} \frac{\cos(x)-1}{x} = \lim_{x\to 0} \frac{1}{x}\left(-1 + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}\right) = \lim_{x\to 0} \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n)!} = 0$

b. $\lim_{x\to 0} \frac{\tan(x)-x}{x^3} = \lim_{x\to 0} \frac{1}{x^3}\left(-x + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)}\right)$

$= \lim_{x\to\infty} \left(\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)\, x^3} \right) = \lim_{x\to 0} \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-2}}{(2n+1)} = \frac{-1}{3}$

7) $\int_0^1 \cos(x^2)\,dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n}}{(2n!)}\,dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}$

$= \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(2n)!\,(4n+1)} \right]_0^1$

First few terms: $\underbrace{1 - \frac{1}{2\cdot 5} + \frac{1}{24\cdot 9}} - \frac{1}{6!\cdot 13} + \dots$

Approximate value $1 - \frac{1}{10} + \frac{1}{216}$ with error $\leq \frac{1}{720\cdot 13} < \frac{1}{1000}$

by alternating series bound for error